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1990 J. Phys. A: Math. Gen. 23 1457

(http://iopscience.iop.org/0305-4470/23/8/019)

J. Phys. A: Math. Gen. 23 (1990) 1457-1463. Printed in the UK

COMMENT

Invertible point transformations, Painlevé analysis and anharmonic oscillators

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Received 20 November 1989

Abstract. The techniques of an invertible point transformation and the Painlevé analysis can be used to construct integrable ordinary differential equations. We compare both techniques for anharmonic oscillators.

For nonlinear ordinary and partial differential equations the general solution usually cannot be given explicitly. It is desirable to have an approach to find out whether a given nonlinear differential equation can explicitly be solved. For ordinary differential equations the Painlevé analysis (see Steeb and Euler 1988 and references therein, Euler *et al* 1989) and the invertible point transformation (Leach and Mahomed 1985, Sarlet *et al* 1987, Duarte *et al* 1987, 1989) can be used to construct integrable nonlinear equations.

In this comment we compare the two methods for anharmonic oscillators.

It is well known that the anharmonic oscillator

$$\frac{d^2U}{dT^2}+U^3=0 \tag{1}$$

can be solved in terms of Jacobi elliptic functions. Let $U_1=U$ and $U_2=dU/dT$; we obtain the autonomous system

$$\frac{dU_1}{dT}=U_2 \qquad \frac{dU_2}{dT}=-U_1^3 \tag{2}$$

with the first integral

$$I(U_1, U_2)=\tfrac{1}{2}U_2^2+\tfrac{1}{4}U_1^4. \tag{3}$$

Then the solution of system (2) can be expressed in Jacobi elliptic functions

$$\begin{aligned} U_1(T)&=\frac{(U_{20}/\omega)\,\mathrm{sn}(\omega T,\mathrm{i})+U_{10}\,\mathrm{cn}(\omega T,\mathrm{i})\,\mathrm{dn}(\omega T,\mathrm{i})}{1+(U_{10}/V)^2\,\mathrm{sn}^2(\omega T,\mathrm{i})} \\ U_2(T)&=\frac{U_{20}\,\mathrm{cn}(\omega T,\mathrm{i})\,\mathrm{dn}(\omega T,\mathrm{i})[1-(U_{10}/V)^2\,\mathrm{sn}^2(\omega T,\mathrm{i})]}{[1+(U_{10}/V)^2\,\mathrm{sn}^2(\omega T,\mathrm{i})]^2} \\ &\quad-\frac{2\omega U_{10}(U_{10}/V)^2\,\mathrm{sn}(\omega T,\mathrm{i})[1+\mathrm{sn}^2(\omega T,\mathrm{i})]}{[1+(U_{10}/V)^2\,\mathrm{sn}^2(\omega T,\mathrm{i})]^2} \end{aligned} \tag{4}$$

where

$$\omega = (\tfrac{1}{2})^{1/2} V \qquad V = (4E)^{1/4} \tag{5}$$

and

$$E = \frac{U_{20}^2}{2} + \frac{U_{10}^4}{4} = I(U_1, U_2) \tag{6}$$

with $i = \sqrt{-1}$ and sn, cn, dn are the Jacobi elliptic sine, cosine, and delta functions respectively. The initial values are $U_{10} = U_1(T=0)$ and $U_{20} = U_2(T=0)$. On inspection we see that (1) admits the two symmetry generators

$$S_1 = \frac{\partial}{\partial T} \qquad S_2 = U\frac{\partial}{\partial U} - T\frac{\partial}{\partial T}.$$

Let us first discuss the Painlevé test. Euler *et al* (1989) studied the anharmonic oscillator

$$\frac{d^2 u}{dt^2} + f_1(t)\frac{du}{dt} + f_2(t)u + f_3(t)u^3 = 0 \tag{7}$$

where f_1, f_2 and f_3 are smooth functions of t with the help of the Painlevé test. We assume that $f_3 \neq 0$. For arbitrary functions f_1, f_2, and f_3 the nonlinear equation (7) cannot explicitly be solved. A remark is in order for applying the Painlevé test for non-autonomous systems. The coefficients that depend on the independent variable must themselves be expanded in terms of t. If non-autonomous terms enter the equation at a lower order than the dominant balance the above-mentioned expansion turns out to be unnecessary whereas if the non-autonomous terms are at dominant balance level they must be expanded with respect to t. Obviously f_1, f_2 and f_3 do not enter the expansion at dominant level.

Euler *et al* (1989) gave the condition that (7) passes the Painlevé test. The condition is given by the differential equation

$$\begin{aligned}9f_3^{(4)}f_3^3 &- 54f_3^{(3)}f_3'f_3^2 + 18f_3^{(3)}f_3^3f_1 - 36(f_3'')^2f_3^2 + 192f_3''(f_3')^2f_3 - 78f_3''f_3'f_3^2f_1 + 36f_3''f_3^3f_2\\ &+ 3f_3''f_3^3f_1^2 - 112(f_3')^4 + 64(f_3')^3f_3f_1 + 6(f_3')^2f_1'f_3^2 - 72(f_3')^2f_3^2f_2 + 90f_3'f_2'f_3^3\\ &- 27f_3'f_1''f_3^3 - 57f_3'f_1'f_3^3f_1 + 72f_3'f_3^3f_2f_1 - 14f_3'f_3^3f_1^3 - 54f_2''f_3^4 - 90f_2'f_3^4f_1\\ &+ 18f_1^{(3)}f_3^4 + 54f_1''f_3^4f_1 + 36(f_1')^2f_3^4 - 36f_1'f_3^4f_2\\ &+ 60f_1'f_3^4f_1^2 - 36f_3^4f_2f_1^2 + 8f_3^4f_1^4 = 0\end{aligned} \tag{8}$$

where $f' \equiv df/dt$ and $f^{(4)} \equiv f'''' \equiv d^4f/dt^4$. It is obvious that we cannot give the general solution to (8). Thus we discuss special cases (Euler *et al* 1989). We recall these cases here because we discuss them in connection with the invertible point transformation.

Case I. Let $f_1(t) = c_1$, $f_2(t) = c_2$, and $f_3(t) = c_3$, where c_1, c_2, and c_3 are constants ($c_3 \neq 0$). Then we obtain from condition (8)

$$c_3^4 c_1^2(2c_1^2 - 9c_2) = 0. \tag{9}$$

Case II. Let $f_1(t) = 0$ and $f_3(t) = 1$. Then we find

$$f_2'' = 0. \tag{10}$$

The general solution is given by $f_2(t) = At + B$, where A and B are the constants of integration. Now (7) takes the form

$$\frac{d^2u}{dt^2} + (A + Bt)u + u^3 = 0. \tag{11}$$

This is a special case of the second Painlevé transcendent. The solutions have no branch points, and are therefore uniform functions in t (see Ince 1956, Davis 1962).

Case III. Let $f_2(t) = 0$ and $f_3(t) = 1$. Then (8) takes the form

$$f_1''' + 3f_1''f_1 + 2(f_1')^2 + \tfrac{10}{3}f_1'(f_1)^2 + \tfrac{4}{9}f_1^4 = 0. \tag{12}$$

This equation admits the particular solutions

$$f_1(t) = \frac{3}{t} \qquad f_1(t) = \frac{3}{2t}. \tag{13}$$

Thus (12) admits more than one branch in the Painlevé analysis. Equation (12) does not pass the Painlevé test because it admits non-integer resonances (rational resonances). However, (12) passes the so-called weak Painlevé test (see Steeb and Euler 1988 and references therein). Curiously these same particular solutions were reported by Moreira (1984) and Leach (1985) studying the modified Emden equation with the direct method for the identification of invariants.

Case IV. A case where f_1, f_2 and f_3 are non-constant and satisfy (8) is given by

$$f_1(t) = \frac{1}{4t} \qquad f_2(t) = \frac{1}{8t^2} \qquad f_3(t) = \frac{1}{32t^2}. \tag{14}$$

Equation (7) together with the functions given by (14) arises in the Painlevé analysis of external driven anharmonic oscillators (Fournier *et al* 1988). This differential equation can be integrated exactly in terms of elliptic functions.

Case V. Equation (7) together with

$$f_1(t) = \frac{1}{4t} \qquad f_2(t) = \frac{1}{8t^2} \qquad f_3(t) = -\frac{1}{8t^2} \tag{15}$$

occurs in the Painlevé analysis of the Lorenz model (Tabor and Weiss 1981). The functions f_1, f_2, and f_3 satisfy (8). Then (7) together with the functions given by (15) can be solved in terms of elliptic functions as follows. Applying the transformation $u(t) = t^{1/4}g(t^{1/4})$ to (7) where f_1, f_2 and f_3 are given by (15) yields $d^2g/ds^2 = 2g^3$ with $s = t^{1/4}$.

Case VI. The equation

$$\frac{d^2u}{dt^2} + \frac{2}{t}\frac{du}{dt} + \frac{1}{t}u^3 = 0 \tag{16}$$

arises in the group theoretical reduction of a nonlinear wave equation. Consequently, we have $f_1(t) = 2/t$, $f_2(t) = 0$ and $f_3(t) = 1/t$. These functions satisfy (8). Therefore, (16) passes the Painlevé test.

Now we ask whether the equation derived above can be found from (1) with the help of the invertible point transformation. Our invertible transformation is given by

$$T(u(t), t) = G(u(t), t) \qquad U[T(u(t), t)] = F(u(t), t) \tag{17}$$

where

$$\Delta \equiv \frac{\partial G}{\partial t}\frac{\partial F}{\partial u} - \frac{\partial G}{\partial u}\frac{\partial F}{\partial t} \neq 0. \tag{18}$$

Since

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\mathrm{d}U}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}U}{\mathrm{d}T}\left(\frac{\partial T}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial T}{\partial t}\right) = \frac{\partial F}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial F}{\partial t} \tag{19}$$

and

$$\frac{\mathrm{d}^2U}{\mathrm{d}t^2} = \frac{\mathrm{d}^2U}{\mathrm{d}T^2}\frac{\mathrm{d}T}{\mathrm{d}t}\left(\frac{\partial T}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial T}{\mathrm{d}t}\right) + \frac{\mathrm{d}U}{\mathrm{d}T}\left(\frac{\partial^2 T}{\partial u\partial t}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial^2 T}{\partial u^2}\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + \frac{\partial T}{\partial u}\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \frac{\partial^2 T}{\partial t^2} + \frac{\partial^2 T}{\partial t\partial u}\frac{\mathrm{d}u}{\mathrm{d}t}\right)$$
$$= \frac{\partial^2 F}{\partial u\partial t}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial^2 F}{\partial u^2}\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + \frac{\partial F}{\partial u}\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \frac{\partial^2 F}{\partial u\partial t}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial^2 F}{\partial t^2} \tag{20}$$

we obtain from equation (1)

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \Lambda_3\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^3 + \Lambda_2\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + \Lambda_1\frac{\mathrm{d}u}{\mathrm{d}t} + \Lambda_0 = 0 \tag{21}$$

where

$$\begin{aligned}
\Lambda_3 &= \left(\frac{\partial G}{\partial u}\frac{\partial^2 F}{\partial u^2} - \frac{\partial F}{\partial u}\frac{\partial^2 G}{\partial u^2} + \left(\frac{\partial G}{\partial u}\right)^3 F^3\right)\Delta^{-1} \\
\Lambda_2 &= \left(\frac{\partial G}{\partial t}\frac{\partial^2 F}{\partial u^2} + 2\frac{\partial^2 F}{\partial u\partial t}\frac{\partial G}{\partial u} - 2\frac{\partial F}{\partial u}\frac{\partial^2 G}{\partial u\partial t} - \frac{\partial F}{\partial t}\frac{\partial^2 G}{\partial u^2} + 3\frac{\partial G}{\partial t}\left(\frac{\partial G}{\partial u}\right)^2 F^3\right)\Delta^{-1} \\
\Lambda_1 &= \left(\frac{\partial G}{\partial u}\frac{\partial^2 F}{\partial t^2} + 2\frac{\partial G}{\partial t}\frac{\partial^2 F}{\partial u\partial t} - 2\frac{\partial F}{\partial t}\frac{\partial^2 G}{\partial u\partial t} - \frac{\partial F}{\partial u}\frac{\partial^2 G}{\partial t^2} + 3\frac{\partial G}{\partial u}\left(\frac{\partial G}{\partial t}\right)^2 F^3\right)\Delta^{-1} \\
\Lambda_0 &= \left(\frac{\partial G}{\partial t}\frac{\partial^2 F}{\partial t^2} - \frac{\partial F}{\partial t}\frac{\partial^2 G}{\partial t^2} + \left(\frac{\partial G}{\partial t}\right)^3 F^3\right)\Delta^{-1}.
\end{aligned} \tag{22}$$

We are not able to handle this general case. We make a particular choice for F and G, namely

$$\begin{aligned}
F(u(t), t) &= f(t)u(t) \\
G(u(t), t) &= g(t).
\end{aligned} \tag{23}$$

With this special ansatz we find that

$$\Lambda_3 = \Lambda_2 = 0 \tag{24}$$

and

$$\begin{aligned}
\Lambda_1 &= \frac{2\ddot{g}\dot{f} - f\dddot{g}}{\dot{g}f} \\
\Lambda_0 &= \frac{(\dot{g}\ddot{f} - \dot{f}\ddot{g})u + (\dot{g}f)^3u^3}{\dot{g}f}
\end{aligned} \tag{25}$$

where $\dot{f} = df/dt$. It follows that

$$\frac{d^2u}{dt^2} + f_1(t)\frac{du}{dt} + f_2(t)u + f_3(t)u^3 = 0 \tag{26}$$

where

$$f_1 = \frac{2\dot{g}\dot{f} - f\ddot{g}}{\dot{g}f} \tag{27a}$$

$$f_2 = \frac{\dot{g}\ddot{f} - \dot{f}\ddot{g}}{\dot{g}f} \tag{27b}$$

$$f_3 = (\dot{g}f)^2. \tag{27c}$$

Here f and g are arbitrary functions of t. We are now able to eliminate f and g from system (27). We obtain

$$g(t) = \int \frac{f_3^{1/2}}{f}\,dt \tag{28a}$$

$$f(t) = Cf_3^{1/6}\exp\left\{\int \frac{f_1(t)\,dt}{3}\right\} \tag{28b}$$

$$f_2(t) = \frac{\dot{f}_1}{3} + \frac{2}{9}f_1^2 + \frac{f_1\dot{f}_3}{18f_3} - \frac{7}{36}\frac{(\dot{f}_3)^2}{f_3^2} + \frac{1}{6}\frac{\ddot{f}_3}{f_3}. \tag{28c}$$

To summarise: equation (1) is transformed into (26) under (17) where

$$\begin{aligned} F(u(t), t) &= \left(C\exp\left\{\frac{\int f_1(t)\,dt}{3}\right\}f_3^{1/6}\right)u(t) \\ G(u(t), t) &= \int \frac{f_3^{1/2}}{f}\,dt \end{aligned} \tag{29}$$

with f_2 satisfying equation (28c) and f_1 and f_3 are arbitrary functions of t.

Let us now compare the two approaches. When we insert (28c) into (8) we find that (8) is satisfied identically. Here we used computer algebra. Thus the invertible point transformations with the special ansatz (23) are a special case of the Painlevé approach. Let us now look for the special cases which we discussed for the Painlevé approach. We find that some cases given in the Painlevé analysis cannot be found with the invertible point transformation.

Case I. Let $f_1(t) = c_1$, $f_2(t) = c_2$. Then we find

$$f(t) = -e^{c_1 t/3} \qquad g(t) = \frac{3}{c_1}e^{-c_1 t/3}. \tag{30}$$

Consequently, the transformation

$$F(u(t), t) = -u(t)\,e^{c_1 t} \qquad T(u(t), t) = \frac{3}{c_1}e^{c_1 t/3} \tag{31}$$

transforms (1) into

$$\frac{d^2u}{dt^2} + c_1\frac{du}{dt} + c_2u + u^3 = 0 \tag{32}$$

with $2c_1^2 = 9c_2$.

Case II. Let $f_1(t)=0, f_3(t)=1$. Then from the invertible point transformation approach we find $f_2(t)=0$. Thus we only find a special case. By calculating the Lie symmetries of (11) we find that if A, $B \neq 0$ the equation has no symmetry generator. If $B=0$ and $A \neq 0$ we find that (11) has the unique symmetry generator $S=\partial/\partial t$. Therefore (11) cannot be obtained from (1) by applying any invertible point transformation. It can be shown that the equations related by an invertible point transformation have the same group structure. Therefore all the equations equivalent to (1) by an invertible point transformation must have two symmetry generators with the same Lie algebra, i.e. $[S_1, S_2] = -S_1$.

Case III. Let $f_2(t)=0$ and $f_3(t)=1$. Then

$$\dot{f}_1 + \tfrac{2}{3} f_1^2 = 0. \tag{33}$$

The solution is given by

$$f_1(t) = \frac{1}{2t/3 + c_1}. \tag{34}$$

This is a special case of (12). We mention that the particular solution $f_1(t)=3/t$ of equation (12) has only one symmetry generator, namely $S=-t\partial/\partial t + x\partial/\partial x$ and therefore cannot be obtained from (1) by an invertible point transformation.

Case IV and V. If we put $f_1(t)=\alpha t^n$ and $f_3(t)=\beta t^m$ we arrive at

$$f_2(t) = (6n+m)\frac{\alpha t^{n-1}}{18} + \frac{2}{9}\alpha^2 t^{2n} - \left(1+\frac{m}{6}\right)\frac{m}{6t^2}. \tag{35}$$

When we set $\alpha=\frac{1}{4}$, $n=-1$ and $m=-2$ we find

$$f_1(t) = \frac{1}{4t} \qquad f_2(t) = \frac{1}{8t^2} \qquad f_3(t) = \frac{\beta}{t^2}. \tag{36}$$

We find f_3 of (14) by setting $\beta = \frac{1}{32}$ and f_3 of (15) by setting $\beta = -\frac{1}{8}$.

Case VI. The functions

$$f_1(t) = \frac{2}{t} \qquad f_2(t) = 0 \qquad f_3(t) = \frac{1}{t} \tag{37}$$

are not solutions of equation (28*c*). We could conjecture that this is related to the fact that we have used the special ansatz (23). That this is not so can be demonstrated by determining the symmetry generators of (26), with f_1, f_2 and f_3 satisfying (37). We only find the symmetry generator $S=-2t\partial/\partial t + \partial/\partial x$. We conclude that this case cannot be obtained form (1) with an invertible point transformation.

If we want to find the first integrals for the equations discussed above we can start from the first integral (3). Using the transformation (23) the first integrals have the form

$$I(u(t), t) = \frac{(f\dot{u} + \dot{f}u)^2}{2\dot{g}^2} + \frac{f^4 u^4}{4} \tag{38}$$

for f_2 satisfying (28*c*), and f and g given by (28*a*) and (28*b*), respectively.

Three remarks are in order. Equation (1) can be considered in the complex plane. Then the singularities are poles of order one. The singularities form a rectangular lattice. Applying the invertible point transformation and using solution (4) we can now study the pattern of the singularities of the transformed equation.

The fifty ordinary differential equations of second order of Painlevé type are just representatives of equivalence classes. The group under which the classification is done is given by

$$\begin{aligned} T &= \phi(t) \\ X(T(t)) &= \frac{\psi_1(t)u(t)+\psi_2(t)}{\psi_3(t)u(t)+\psi_4(t)} \end{aligned} \tag{39}$$

where ψ_1, ψ_2, ψ_3, ψ_4 and ϕ are analytic functions of t.

Finally we mention that for certain choices of the parameters, the Painlevé transcendents *II*-*V* admit one-parameter families of solutions expressible in terms of classical transcendental functions, such as Airy, Bessel, Weber-Hermite and Whittaker, respectively (Gromak 1978).

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